

Reply to 'Comment on 'Invariants of differential equations defined by vector fields''

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REPLY

Reply to ‘Comment on ‘Invariants of differential equations defined by vector fields’’

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Abstract

We undertook the determination of fundamental sets of invariants for differential equations defined by vector fields in a recent paper. Using the equivariant moving frame method, a more exhaustive list for these invariants was found in a subsequent comment by Francis Valiquette. We reply to that paper and point out some essential inconsistencies in the comment.

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In [1] we obtained differential invariants of order $p \leq 2$ for the group of equivalence transformations of differential equations defined by vector fields, and which may be put in the form

$$\sum_{i=1}^n a^i(x) \partial_{x_i} u(x) = 0, \quad x = (x^1, \dots, x^n), \quad (1)$$

where u is the dependent variable, and the arbitrary coefficients $a^i(x)$ are generally assumed to be nonzero. More precisely, we showed that there are no zeroth-order invariants, and that for $p = 1$, the fundamental invariants $T_{i,j}$ and their number \mathcal{N} are given for all $n \geq 2$ by

$$T_{i,j} = \frac{a_j^i a^j}{a^i}, \quad (i \neq j), \quad \mathcal{N} = n(n-1), \quad \left(a_j^i = \frac{da^i}{dx^j} \right). \quad (2)$$

We also showed that for $p = 2$, when $n = 2$, the invariants are of type

$$T_{i,j} = \frac{a_j^i a^j}{a^i}, \quad \text{and} \quad K_{ij} = \frac{a_{jj}^i a^j}{a_j^i} + a_j^j, \quad (i \neq j), \quad (3)$$

while for $p = 2$ and for all $n \geq 3$, a fundamental set of invariants was shown to include all of the functions

$$T_{ij}, \quad K_{ij}, \quad \text{and} \quad L_{ijk} = a_{jk}^i \left(\frac{a^j a^k}{a^i} \right), \quad \left(a_{jk}^i = \frac{d^2 a^i}{dx^j dx^k} \right), \quad (4)$$

where $i, j, k \in \{1, \dots, n\}$ are pairwise distinct indices. In a recent comment [2] on the paper, new invariant functions of (1) were found using the equivariant moving frame method. However, a number of remarks are in order concerning this comment.

- (a) The abstract and the introduction of the comment [2] are misleading about the original paper [1] it criticizes, and gives in particular the impression that only the cases $n = 2$ and $n = 3$ are treated in our paper for differential invariants of order up to 2. In reality, equation (2) above simply represents theorem 4 of the original paper that treats the general case for $n \geq 2$ and $p = 1$, and which is in agreement with proposition 4 of the comment for these values of n and p . Equation (4) on the other hand expresses the result of theorem 7 of the original paper which also indicates the number of all invariants in (4) for all $n \geq 3$.
- (b) It is also not correct to claim that the determination of all invariants took only one page in [2] and about seven pages in [1], as most of the discussion in the determination of invariants in the latter paper is aimed at analyzing the Lie algebraic structure of a certain family of vector fields associated with the infinitesimal generator of the equivalence group. It is clearly stated in [3] that moving frame algorithms can be computationally very demanding and tend to require sophisticated computer algebra technology. It can therefore be difficult and futile to try to find out in this specific context which of the methods of [1, 2] is faster.
- (c) That said, the equivariant moving frame method has important advantages because it is not always obvious to find the infinitesimal generators of Lie group actions, and especially of induced Lie group actions, and the moving frame method does not require precisely such types of generators for the determination of invariants. Thanks to this method, a new type of invariants of (1) was obtained in [2], namely invariants of order two of type

$$R_{ij} = a^j a_{ij}^i - \frac{a^j}{a^i} a_j^i a_i^i, \quad (i \neq j), \tag{5}$$

and this is however the only type of invariants found in the comment that does not appear in [1]. Nonetheless, the discovery of this new type of invariants definitely invalidates the incomplete result of theorem 6 of [1], which should have been rather stated as a conjecture, on the basis of certain doubts we had during the preparation of the paper about the validity of the expression

$$\mathcal{V} = \sum_i^n \xi^i \partial_{x^i} + \sum_i^n a^i \xi^{i'} \partial_{a^i} \tag{6}$$

for the infinitesimal generator \mathcal{V} of the equivalence group of (1). The discovery of the new type of invariants also invalidates the conjecture on page 12 of [1], stated only in the hope that there was no invariants of order two left out in our calculations.

- (d) The only cause of the difference between the invariants found in [1, 2] appears to be the incorrect expression for the infinitesimal generator (6), which as indicated in [1], was obtained using the method of [6] of [1]. We have already had the opportunity to signal in another recent paper [4], that the said method often leads to the wrong infinitesimal generator by giving explicit examples. More specifically, we have indicated in [4] that an attempt to find second-order differential invariants using this \mathcal{V} gives rise to functions of the type

$$M_{ijk} = a_{jk}^i \left(\frac{a^j a^k}{a^i} \right)^2, \quad i, j, k \text{ pairwise distinct} \tag{7}$$

which all satisfy the infinitesimal condition of invariance $\mathcal{V} \cdot M_{ijk} = 0$, but which are not actually invariants of (1), while the true invariants L_{ijk} appearing in (4) and obtained in [1] using a different technique do not satisfy the required condition $\mathcal{V} \cdot L_{ijk} = 0$. This also means that we have already reported in [4], either explicitly or tacitly the essential criticisms in the comment about our paper. Naturally the new invariants of type (5) also do not satisfy the infinitesimal condition using the generator \mathcal{V} .

- (e) In fact, it is precisely because we discovered the invalidity of the infinitesimal generator (6) derived in [1] using the cited method that we undertook for the determination of another method for finding these infinitesimal generators. The method thus obtained in [4] works well for all linear ODEs, and a variety of PDEs and nonlinear ODEs that we have tested. We must however recall at this point that the same infinitesimal generator (6) used above in [2] for the classification of differential invariants might lead to wrong results.

In conclusion, the equivariant moving frame method has proved to be advantageous in the determination of invariants of differential equations based on a Lie group approach. However, type R_{ij} invariants (5) is the only new type of invariants found in the comment, and the difference between the invariants found in [1, 2] is only due to the incorrect expression for the infinitesimal generator \mathcal{V} in (6), which was derived in [1] using the method indicated in that paper. Incidentally, we have already reported in [4] this incorrectness of \mathcal{V} together with typical errors it generated in the original paper.

References

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